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N 73 - 19433

NASA CR-112259

ACTIVE OPTICS SIMULATION SYSTEM

By Chang H. Chi

January 1973

Prepared Under Contract No. NAS 1-11205 by
THE PERKIN-ELMER CORPORATION
Norwalk, Connecticut
for
NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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FOREWORD

This final report on the Active Optics Simulation System (AOSS) is one of three documents generated under NASA Contract No. NAS 1-11205, sponsored by NASA, Langley Research Center. The other documents are the User's Manual, Perkin-Elmer Report No. 11315, and the Programmer's Manual, Perkin-Elmer Report No. 11397.

The User's Manual contains the necessary information the user will need to operate AOSS and the Programmer's Manual contains the details of AOSS programming.

The final report is the summary of the entire AOSS project effort. It describes AOSS and the role it can play in the development of large space optical systems, the stage of development that AOSS has reached, and the results obtained using AOSS.

This report will be sufficient for those who want an overall view of the project but do not plan to use AOSS in actual practice. For those who plan to use AOSS and wish to obtain the details of actual implementation, the User's Manual is the primary document to be consulted.

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ACTIVE OPTICS SIMULATION SYSTEM

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SUMMARY

The Active Optics Simulation System (AOSS) is a set of computer programs and associated software to be used in the development, design, and evaluation of a primary mirror control system for a large space telescope (e. g., the tentatively proposed 3-meter telescope). The mathematical models of component subsystems and the solution of the physical processes that occur within the mirror surface control system have been obtained, and based on these models AOSS simulates the behavior of the entire mirror surface control system as well as the behavior of the component subsystems.

The program has a modular structure so that any subsystem module can be replaced or modified with minimum disruption of the rest of the simulation program.

This simulation system is developed to be useful to the engineers who are developing the mirror surface control system, and, after the usual "field service phase", it will become a valuable design aid for them in the various phases of development, design, and optimization of design parameters. Therefore, the simulation system incorporates many user-oriented convenience features to free the engineer from tedious programming details and to allow him to concentrate his effort on engineering development.

INTRODUCTION

In the field of astronomy and deep space communications, the large telescope orbiting outside the earth's atmosphere provides a manifold increase in the amount and resolution of scientific data. With increasing primary mirror diameter comes higher image resolution and optical power. The resulting higher signal-to-noise ratio permits meeting the requirement for more accurate pointing and stabilization. The larger primary mirror, on the other hand, requires higher surface accuracy ($\lambda/50$ or better) in order to obtain its higher limiting image resolution. The design, fabrication, and testing of such mirrors must consider mirror surface deformation due to gravity release, thermal gradients and loading, and any other disturbances that will cause surface changes in orbit.

The question arises whether the approach of controlling the primary mirror surface by the feedback control technique is feasible and offers overall advantages. This approach has the inherent capability to achieve an accurate mirror surface figure in the presence of any type of disturbance including the gravity release and thermal loading, although extra hardware is required to implement the control system.

The active optics technology has in the past demonstrated the feasibility of the technique through a working laboratory model (Refs. 1 and 2), and has entered the development phase in which some advanced control concepts are being investigated (Refs. 3 and 4) and improvement is being made to various subsystems.

Work covered under the present project has included the establishment of the mathematical model of the physical hardware and mathematical description of the physical processes, conversion of these mathematical formulations into the form compatible with efficient computation and programming, and the generation of the computer programs and the necessary documentation.

A smaller number of test cases have been generated to test the working of AOSS in this project. However, the scope of the present project does not include the investigation of the active mirror control system itself using AOSS. As stated earlier, these tasks will be performed during the "field service phase", in which AOSS will be modified, extended and, in general, improved as experience is gained.

The development of the Active Optics Simulation System has been initiated in the belief that it can be a useful tool and a development aid in the current and future phases of the active optics technology development.

OBJECTIVE AND SCOPE OF THE ACTIVE OPTICS SIMULATION SYSTEM (AOSS)

In the development of the active mirror control system a variety of control concepts is proposed; these concepts include the simple proportionality control scheme, the recently proposed modal control scheme, and other modified modal control schemes. The behavior of some of these systems is usually so complex that any meaningful analysis or evaluation of a particular system has not been possible beyond the level of very simplified and crude guesses, if at all.

The large mirror control system can assume a variety of system configurations: a large thin or thick deformable mirror, a large segmented mirror, a system with fixed and/or movable supports, and others that are known at present and will be proposed in future. Each subsystem that the mirror control system comprises can also be one of many types of configurations or concepts; different types of figure sensors, control laws, actuators, support modes, backing plates, mirror shapes, and disturbances. When designing a particular system, a way must be available to predict the behavior of the system with adequate accuracy so that the system parameters can be optimized.

The development of the Active Optics Simulation System (AOSS) was initiated in order to simplify the enormous calculation tasks associated with the analysis and optimization of the active optics mirror control system. The AOSS is a set of computer programs and associated software that simulate by computer calculation the behavior of the entire mirror surface control system as well as the behavior of the component subsystems. It can be used by an engineering team to evaluate the merit of a particular active optics control system (or a component subsystem); and, once the system configuration is chosen, can be used as a design aid to achieve the optimization of the system parameters.

We believe that AOSS will be an indispensable design tool when the design of the active optics control system actually begins and when we need answers to some basic questions; for example, given a set of mirror surface figure requirements, "How many actuators do we need?" "Where should the control actuators be placed?" "How much improvement in mirror surface figure do we gain if one extra actuator is added?"

Since AOSS is intended to be an engineering tool, its merit can only be determined in terms of the degree to which it represents a simplification of the engineer's work.

The objective of the present AOSS project was to establish the basic framework of the simulation system and to produce working computer simulation software.

AOSS is expected to go through the usual "field service phase" in which it will be modified, improved, and extended to better meet the needs and convenience of the users, eventually becoming an important "design tool" to those working on the mirror control system. Therefore, there are two major objectives to keep in mind as the future AOSS projects are contemplated: One is that AOSS, while passing through the process of continuous modifications, should at the same time serve as a research and development tool to the user in the development stage of the mirror control system; the other is that a well-developed and well-tested AOSS should be ready and available when the user is called upon to produce the optimum design of the mirror control system for the large space telescope (LST) or any other applications.

The mathematical models for AOSS are intended to describe the particular subsystems accurately, incorporating enough detail so that the computed results are a reliable and realistic representation of actual response, and omitting needless details so that the complexity of the programming and computer time is kept to the manageable level. When developing AOSS, a considerable effort was made to anticipate the type of information required and the sequence of the most frequently performed operations, and to construct AOSS accordingly. (See Ref. 5.) It was recognized at the beginning of the AOSS project that the best effort should be devoted to produce the optimum design in terms of the necessary accuracy and reliability of the computer results, the amount of computer time, and the programming complexities. Consequently, we have developed some new methods and mathematical formulation specifically suited to AOSS. We have designed a new polar grid pattern for the circular geometry with point supports on the circumference and an accompanying grid point numbering system. We have also developed an interpolation scheme, called the curvilinear bicubic spline fit interpolation, and the quasi-optics formulation which includes the dominant term of the diffraction effects as well as the geometric optics. These developments are described in detail in the User's Manual (Ref. 6).

Features of AOSS

The simulation system has been developed with three prominent features in mind: modular structure, engineer-oriented convenience features, and an "interactive mode" of operation.

Modular structure. - The program is composed of several physically or functionally distinct program modules, such as mirror module, backing plate

module, figure sensor module, control law module, actuator module, disturbance module, interpolation module, and others (see section on "Structure of the Simulation System").

This feature allows each program module to be replaced or modified without disturbing the rest of the system unduly. As the technology progresses, different mirror control system configurations or new control concepts (i.e., new control laws, or figure sensor, mirror, or actuator configurations, etc.) can be investigated by creating the module that corresponds to the particular subsystem and incorporating it into the Active Optics Simulation Program.

Engineer-oriented convenience features. - The simulation system includes a number of convenience features. These features relieve the engineer of unnecessary chores so that he can concentrate on the design of the control system. Some examples are described below:

(a) The program contains the mirror material characteristics data shown in Table 1. The engineer writes, for example, type "8" in order to specify fused silica and need not bother to enter all the material constants.

(b) The program contains five types of mirror mounting modes. When the engineer wishes to use three-point kinematic support systems, he needs only to specify, for example, type "2" with the mount locations. The program computes all the necessary boundary conditions.

(c) In designing the control compensators the engineer needs only to specify the time constants and gain, and the simulation system converts them into the appropriate coefficients of the differential equations.

(d) Although not fully implemented at this time, AOSS is structured to accommodate, without any difficulty, input and output programs that allow the engineer to input and output data in terms of engineering quantities such as optical path difference or f-number (optical engineering), phase margin and break frequencies (control engineering), time constants and equivalent circuit parameters (electrical engineering), and stress and forces (structural engineering).

AOSS places a great deal of emphasis (and this emphasis will continue in the future development) on the features that are helpful to the engineer. Often, the program complexity and computer time are sacrificed in order to provide

TABLE 1. - MATERIALS AND THEIR PROPERTIES PROVIDED IN THE AOSP AUXILIARY PROGRAM

Material	MID	E Newtons/m^2	G Newtons/m^2	ν	ρ kg/m^3	α_T (m/m)/°C
Cer-Vit 101	3	9.24×10^{10}	3.65×10^{10}	0.25	2.50×10^{3}	0.15×10^{-6}
Cer-Vit 126	4	8.48×10^{10}	3.38×10^{10}	0.26	2.51×10^{3}	0.15×10^{-6}
ULE Corning 7971	5	7.03×10^{10}	3.01×10^{10}	0.17	2.213×10^{3}	0.06×10^{-6}
Corning Fused Silica 7940	6	7.60×10^{10}	3.23×10^{10}	0.17	2.202×10^{3}	0.49×10^{-6}
Zerodur	7	9.39×10^{10}	3.78×10^{10}	0.245	2.52×10^{3}	0.15×10^{-6}
Fused Silica	8	6.89×10^{10}	3.08×10^{10}	0.2	2.2×10^{3}	0.55×10^{-6}
Beryllium	9	28.0×10^{10}	13.67×10^{10}	0.024	1.82×10^{3}	12.4×10^{-6}
Invar (36% Ni)	10	14.48×10^{10}	5.59×10^{10}	0.29	8.13×10^{3}	1.26×10^{-6}
Invar (42% Ni)	11	15.17×10^{10}	5.86×10^{10}	0.29	8.12×10^{3}	5.72×10^{-6}
Jig Plate Al-Alloy Casting, Alcoa Type 300	12	7.10×10^{10}	2.67×10^{10}	0.33	2.80×10^{3}	23.58×10^{-6}

the engineer with extra benefits in terms of convenience and carefree operation. For example, automatic checks for errors in the input data, which range from the trivial case of a missing (or duplicate) data card to the more complex case of physically inconsistent or meaningless data.

Another example of the user convenience feature occurs in the grid point numbering system. In the structural analysis, the grid point numbering system is chosen judiciously so that the computer solution can be most efficient and accurate. However, the grid point system thus chosen, called the machine-oriented grid point system, causes a great deal of confusion and aggravation when the user examines the printed output of the computer solution; therefore, another set, called the user-oriented grid point system, is generated for practical use. AOSS executes the structural computation using the machine-oriented grid point system and outputs the results in the user-oriented grid point system. (See section on "Auxiliary Program".)

In the future development, AOSS can easily provide diagnostic messages for the error and failure modes, a warning signal when some physical system reaches the limit of its operation such as the range of the mirror elasticity (i.e., mirror is about to break) or the linear range of the electronics system.

"Interactive mode" of operation. - AOSS does not have the capability to operate in an "interactive mode" at this time. However, sufficient thought has been devoted to this feature so that the implementation can proceed rapidly once the specific demand arises.

The "interactive mode" of operation means that the engineer can sit at a remote console that includes a cathode ray tube (CRT) display and a keyboard. The CRT can display either numerical values or a graphical display or both at the same time. During the computation, the engineer can observe the system behavior by displaying a set of quantities of his choice (such as mirror surface error, forces, mode amplitude, etc.) on the CRT, and, if necessary, he can interrupt the computer, enter a set of modified input parameters and continue or restart the computation. This feature allows the engineer to design the control system efficiently and to proceed without interruption.

A typical example is of an engineer who sits at the remote console, enters the number and locations of the actuators, and observes the behavior of the mirror control system. He can change the number and location of actuators and observe the new system behavior. Similar operations can be performed for the figure sensors.

Documentation

The documentation is an important part of the AOSS project, as is true with any software development project. Therefore, a major effort and a considerable amount of time have been devoted to the task of preparing the necessary documentation of AOSS.

It is worth emphasizing that without adequate documentation the computer programs are difficult to use, impossible to modify, and become obsolete in a short time.

It is estimated in the AOSS project that the documentation and the related tasks to facilitate easier understanding of the documentation consumed as much effort and time as did the design and implementation of the simulation system proper, and this serves to demonstrate the magnitude of the effort and the emphasis the documentation has received in the current project. Nevertheless, as stated in the introduction to the User's Manual, the resulting documentation still has areas requiring improved format and updating.

The AOSS project has prepared three documents: the User's Manual (Ref. 6), the Programmer's Manual (Ref. 7), and this final report.

The User's Manual contains the necessary information for the user who will operate AOSS and proceed to investigate and to design the active optics mirror control system. The user is considered to be typically an engineer who is engaged in the development of the mirror surface control system. He is expected to be familiar with the basic theories connected with the mirror surface control technology but is not required to be familiar with the programming aspect of AOSS. The documentation is organized so that the User's Manual is the only document the user will need to operate AOSS and conduct his investigation. The User's Manual includes the operating input/output instructions, sufficiently detailed block diagrams with accompanying explanations to show the user precisely how the sequence of the mathematical operations are carried out within AOSS, and some important theoretical developments upon which AOSS operations are based.

The Programmer's Manual contains the detailed computer programming information as it is prepared for a typical computer programmer who will be called upon to implement any future modification and extension of AOSS and also to diagnose and troubleshoot any programming inconsistencies arising as different types of computers and terminal equipment are interfaced.

The programmer is not required to be familiar with the theoretical basis and the working of the mirror control system, and he will have no need to consult the User's Manual or this final report in any detail, if at all, in performing his tasks.

This final report is the summary of the entire AOSS project effort and presents the overall view of AOSS development. It describes the AOSS structure, the various operations performed within AOSS and the present capabilities in a general terminology easily understood by those who are not intimately familiar with the working of the active optics control system.

This final report also presents the overall view of AOSS development effort and describes how AOSS can contribute and play a part in the development effort of the large space telescope (LST). It evaluates the present stage of AOSS development and anticipates the necessary future development effort for AOSS.

Throughout the entire documentation, MKS system is consistently used; for example, Kg is the unit of mass and not the unit of force nor weight in the structural analysis. Also the cylindrical coordinate system has been consistently used throughout the documentation.

Each program package consists of a number of smaller modular programs, called submodules. Each submodule program has been assigned an alphanumeric code, and this code has been consistently used in all phases of the documentation: in the block diagrams of both the User's Manual and the Programmer's Manual, in the description of submodules, and in the program coding with the use of the comment cards.

The attempt to standardize the mathematical notations and symbols turned out to be one of the most arduous tasks during the documentation. The reason is that each engineering discipline has its own set of symbols whose meaning has long been established and any deviation from these will cause more hardship than any advantages to be gained. For instance, an electrical engineer will have difficulty bringing himself to accept that τ is the shear stress and not the time constant of a transient response. Besides, there are not enough letters in the English and Greek alphabets to satisfy our needs without having too many subscripts or superscripts. Eventually, it was decided that the auxiliary program and the structural analysis will have their own notations and symbols, regardless of what is used in other sections. Thus, a list containing the notations and their meaning was prepared separately to be used for the structural analysis only.

Each of the three documents has been prepared in a self-contained volume so that a minimal amount of cross-referencing among the manuals is required, eliminating much of the user inconvenience and frustration. For this reason, some parts of this report are repetitions of parts of the manuals, but this repetition is done only when there is ample reason to do so. We believe that the manuals should be as simple and concise as possible and that thick manuals neither necessarily mean that "lots of work was performed" in the project, nor represent good documentation.

STRUCTURE OF THE SIMULATION SYSTEM

The active optics simulation system (AOSS) consists of three major computer program packages: Auxiliary Program, NASTRAN Program, and Active Optics Simulation Program (AOSP). There is also a secondary program package, called preprocessing program, which provides the necessary interface, if any, between NASTRAN and AOSP.

The block diagram of the simulation system structure is shown in Figure 1.

The Auxiliary Program receives from the user a set of structural data (mirror, backing plate, support mode) and generates the necessary input card deck (executive control cards, bulk data cards) in the format required by NASTRAN to obtain the structural solutions. During the process of converting the user input data to the NASTRAN input cards, the Auxiliary Program performs some simple error-checking sequences, which check for any obvious errors in the user input data such as missing or duplicate data cards, contradictory instructions, and physically meaningless input data.

The NASTRAN (NASA Structural Analysis) Program is a very large computer program that can produce the structural solutions and perform many other mathematical operations as well (Ref. 8 through 11).

The structural solution can be in the form of eigenfunctions and eigenvalues, stiffness matrix, or influence coefficient matrix. NASTRAN can be made to output any one or all of the above forms of solutions. In addition, NASTRAN can produce the thermal flexibility matrix (B_T), which represents the mirror deformation at all the grid points due to a unit temperature rise at a given grid point.

It is to be noted that any one form of the structural solutions is sufficient and describes the structure completely. However, since one form of the structural solutions is more convenient to use than others in a given calculation, it is useful to have available all the different forms of solution.

In the present simulation system, the eigenfunctions and eigenvalues are outputted, and other forms of the solution can be outputted easily when needed.

The NASTRAN Program outputs the results of the structural analysis and also some of the user input data needed in the later programs. These outputs are stored in the tape or punched card deck. As soon as the NASTRAN outputs are obtained and stored, the NASTRAN Program is disengaged from the simulation program system and taken off the computer.

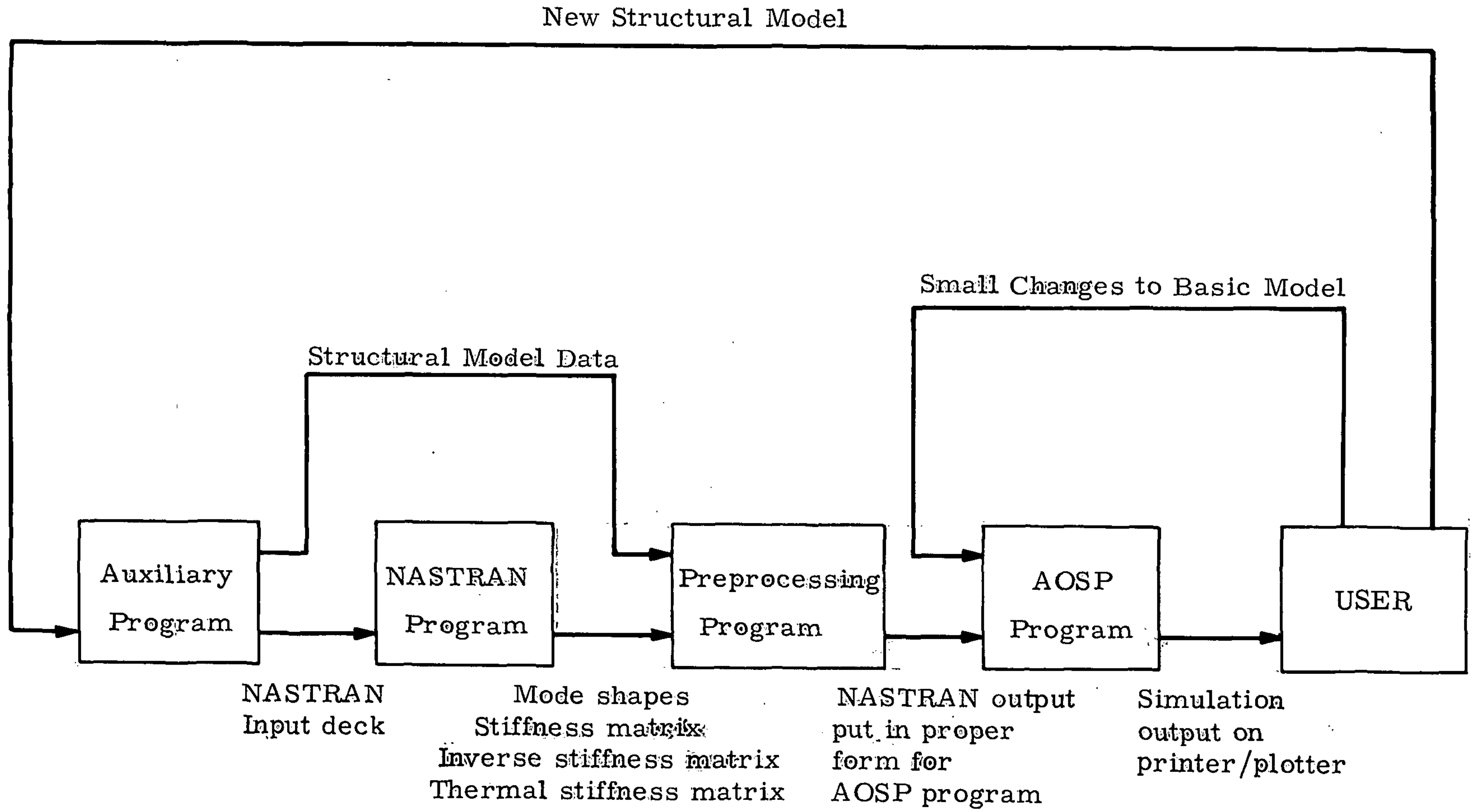

Figure 1. Programs Involved in Active Optics Design Loop

The preprocessing program receives the NASTRAN outputs and makes any necessary modifications and rearrangements to the data. The tasks performed by the preprocessing program will be different for different computer facilities because for each computer facility the size and type of the computer, the characteristics of the terminal display equipments, and the work procedure of that facility will determine the content of the preprocessing programs.

The block diagram of the Active Optics Simulation Program (AOSP) is shown in Figure 2. It is a completely self-sufficient program and does not depend upon the NASTRAN Program or any other program in order to operate.

The AOSP program is composed of a number of program modules; each module represents a physically distinct hardware system or a functionally distinct mathematical operation (as in the case of the interpolation module).

The composite system of the mirror - backing plate - mount mode is represented by the eigenfunctions and eigenvalues of the structure (or some other forms of solution) which NASTRAN produced and is available to the AOSP (see NASTRAN outputs in Figure 1).

The interpolation module represents not a hardware system but a mathematical operation necessary in any computer simulation system. It is to be noted that the computer-calculated values are given at a set of discrete grid points; thus, when a value at the intermediate point between the grid points is needed, it must be obtained by using a suitable interpolation scheme.

The structure of the simulation system is the modular form. This modular approach has been used throughout the system and also within each module. Consequently, many submodules make up a module. The modular structure of the simulation system allows each constituent module to be replaced or modified without disturbing the rest of the system unduly. For instance, the force actuator module can be replaced by the displacement actuator module and the simulation system will continue to operate without any modifications.

It is to be noted that the NASTRAN Program is a large program and capable of performing a wide range of mathematical operations, as well as structural analysis. Many of the operations performed by AOSP can indeed be performed by NASTRAN. However, in many instances, the use of NASTRAN presents difficulties because it takes up too large an amount of the computer core to be reasonably efficient. At Langley Research Center and other facilities, NASTRAN is run at night only. This situation is not compatible with the way AOSS is intended to be used, which is to provide the user with a quick turnaround time and to have the computer available to the user on a continuing basis during the day.

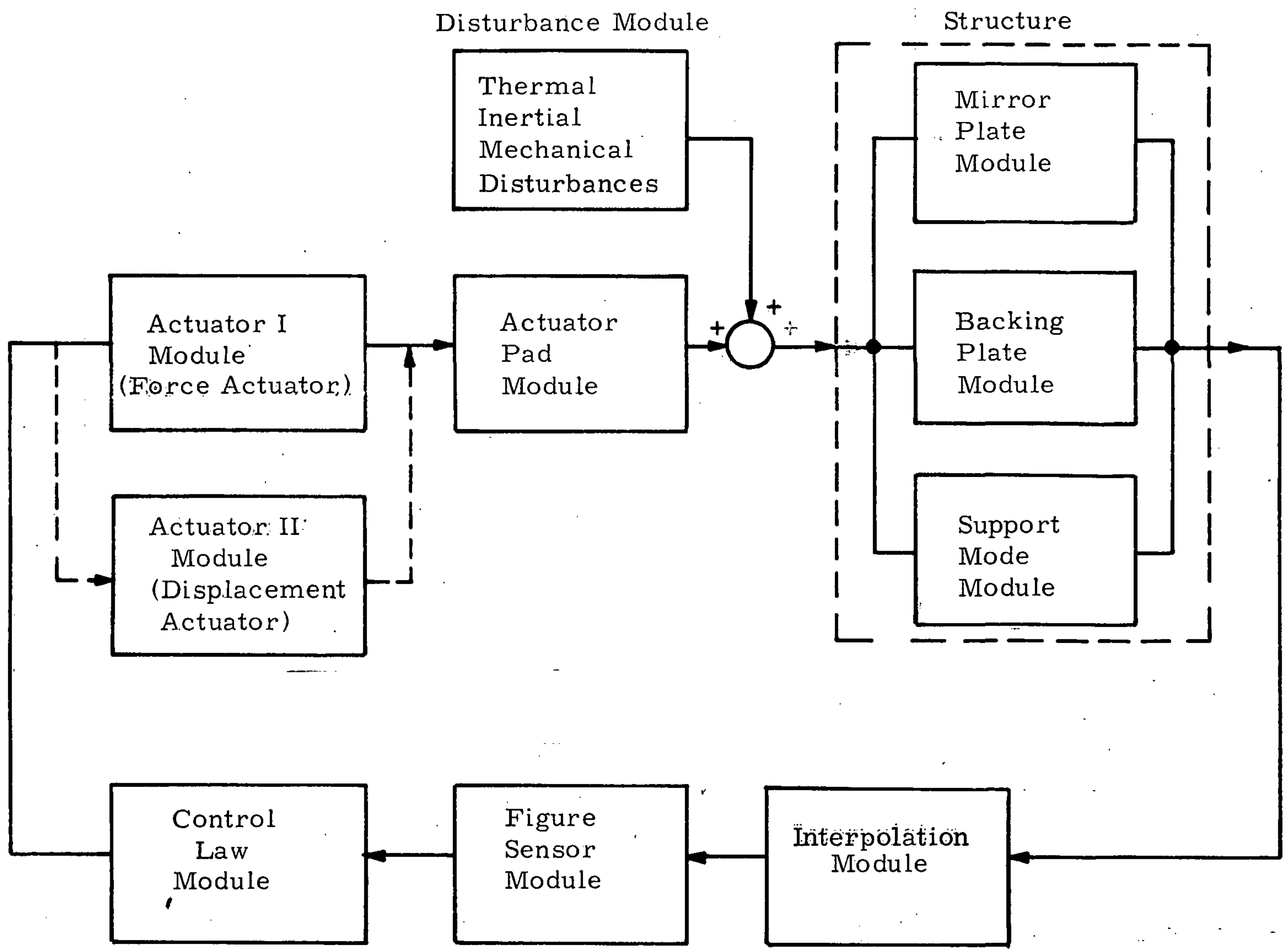

Figure 2. Major Subsystems in AOSP (Active Optics Simulation Program)

For these reasons, the simulation system is divided into two parts, as described earlier, so that the user can work separately with AOSP, which is a relatively small program and can be operated at any time.

The simulation system, structured in such a way, is compatible also with the "interactive mode" of operation envisioned as a future development of the simulation system.

AUXILIARY PROGRAM

The Auxiliary Program receives from the user all the pertinent physical information on the structure of the mirror and backing plate and the support modes. It then generates the necessary input bulk data cards (in the form of either cards or recorded data on magnetic tape) in the order and format required by NASTRAN to produce the structural solutions.

The structural solutions can assume many forms: the eigenfunctions and eigenvalues, the flexibility matrix, and the stiffness matrix. At present, the AOSS Auxiliary Program outputs the set of 139 eigenvalues and 139 eigenfunctions, although the flexibility matrix and the stiffness matrix can easily be outputted when the need arises. The Auxiliary Program also outputs the thermal flexibility matrix, denoted by $[B_T]$, which represents the structural deformation at every grid point due to a unit temperature rise at a given grid point.

It is to be noted that any one form of the above structural solutions completely defines the structural behavior and therefore is sufficient. However, one form of solutions is easier to use than others in a given situation; and, since all forms of the solution are available from NASTRAN, it would be wise to take advantage of NASTRAN capability by providing the option to call for any one or all of the forms of solutions.

In keeping with the user convenience features stressed in AOSS, the Auxiliary Program performs some basic error checking on the input data provided by the user: It catches the missing or duplicate input cards submitted by the user by mistake, and some user input data that are contradictory or make no physical sense. Some prominent user convenience features implemented in the Auxiliary Program have been presented in the section on "Features of AOSS". However, there are many more that are worth providing, including those items that allow the user to specify the structural data in terms of engineering parameters such as f-number, radius of curvature, thickness-to-diameter ratio, etc.

Since AOSS depends on NASTRAN for its structural solutions, the future improvement and refinement of NASTRAN will benefit AOSS to the same degree. AOSS has been developed with this fact in mind so that any future NASTRAN capability can easily be utilized.

It is very important to realize that the computer simulation project is not "feeding the computer with the input data and letting the computer produce the answer". If a set of data is blindly submitted to the computer program without realizing the program's limitations it will almost always result in a grossly wrong result.

Indeed, the major part of the computer simulation project is to make sure that the computer solutions are correct answers having sufficient numerical accuracy and as high a degree of credibility as is possible. Therefore, in the development of AOSS structural analysis, a number of important preparatory tasks had to be accomplished before the computer was actually "turned on" to compute the structural analysis.

Initially, a theoretical analysis* was performed and the results were studied; the analysis obtains the eigenvalues and eigenfunctions (vibrational frequencies and modes) for a flat circular plate simply supported at three points along the circumference.

The theoretical analysis is useful as a means to ascertain the correctness of the computer solutions and evaluate the accuracy of the computer solutions. It is also very important in that it shows what to look for in the large maze of printed numbers output from the computer; without the guide of the analytical results the project easily gets lost and quickly bogs down.

The analysis was carried out on the system of a flat plate with support points shown in Figure 3, and some of the mode shapes and the corresponding normalized eigenvalues are shown in Figures 4 through 11 with Poisson's ratio equal to 0.25.**

The eigenvalue (k) is defined to be

$$k^4 = \frac{\mu}{D} \omega^2 \tag{1}$$

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*The theoretical analysis (Refs. 12 and 13) was initially performed under Contract NAS 1-9759 and the further investigation has continued in the present project.

**The discussions comparing some of the eigenvalues presented in this report and those of the free plate tabulated in Leissa's monograph (Leissa, 1969), p. 11, Table 2.6, are presented in reference 14 (see Table 2 of reference 14).

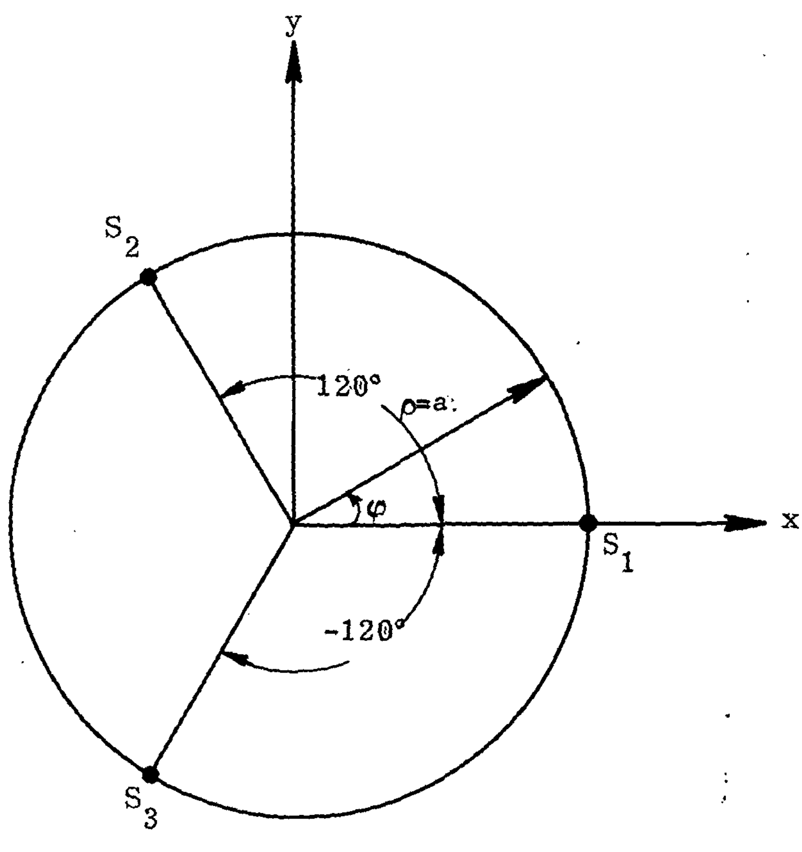


Figure 3. Thin Flat Circular Plate with Three Simple Support Points S_1, S_2, S_3 on the Circumference

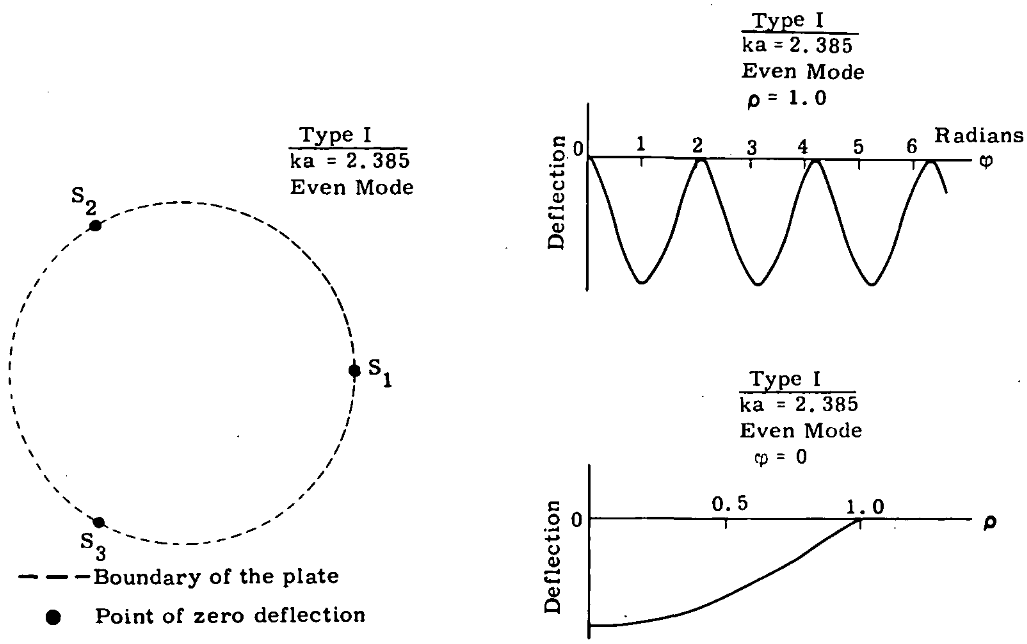


Figure 4. Mode Shape, Type I, ka = 2. 385

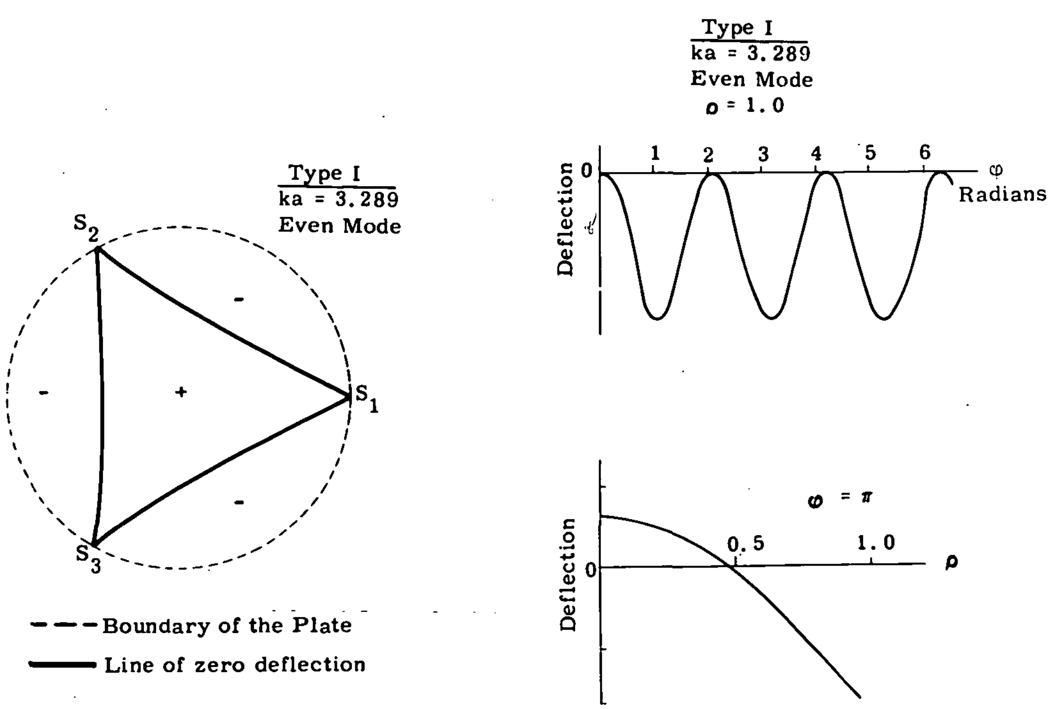


Figure 5. Mode Shape, Type I, ka = 3. 289

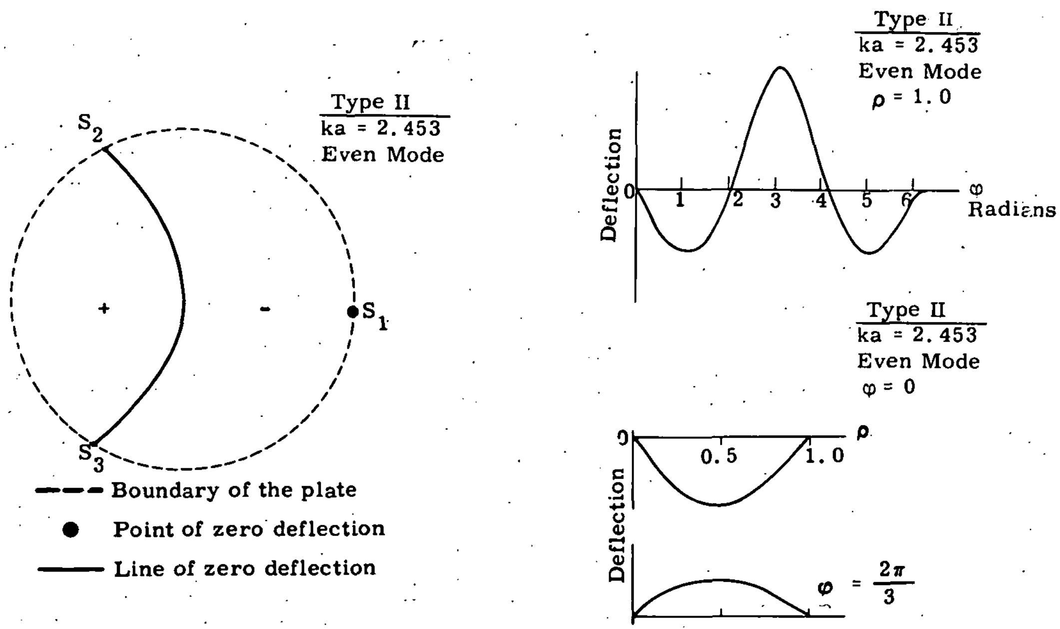


Figure 6. Mode Shape, Type II, ka = 2.453

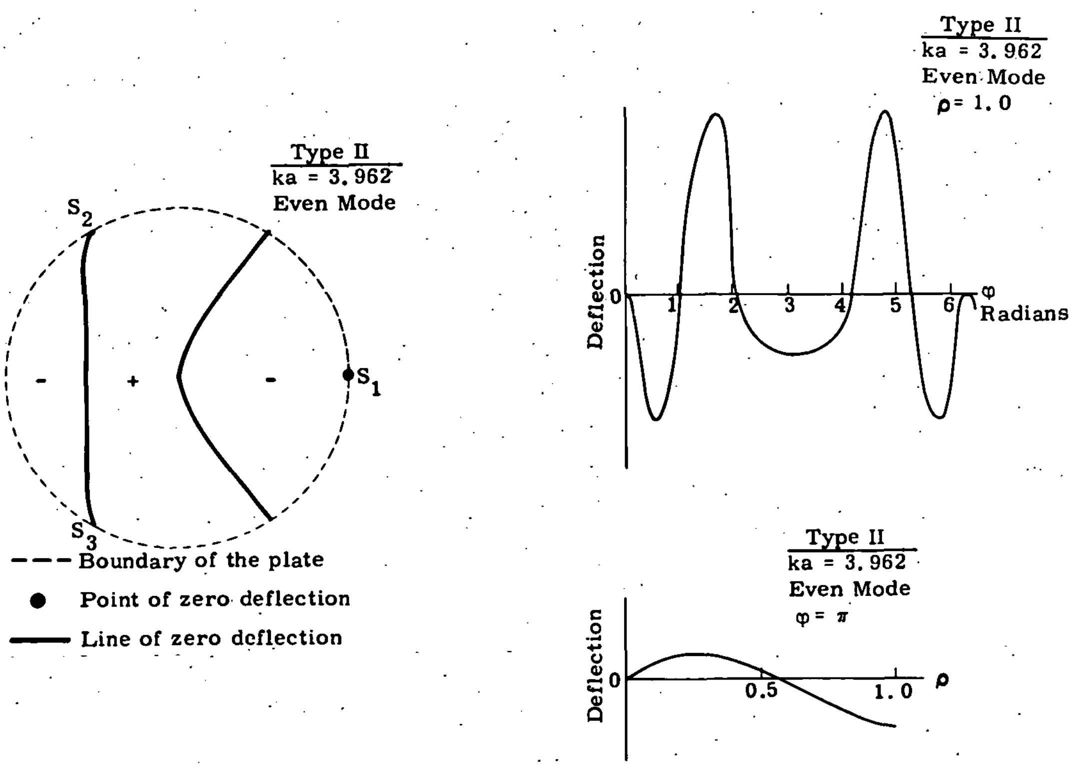


Figure 7. Mode Shape, Type II, ka = 3.962

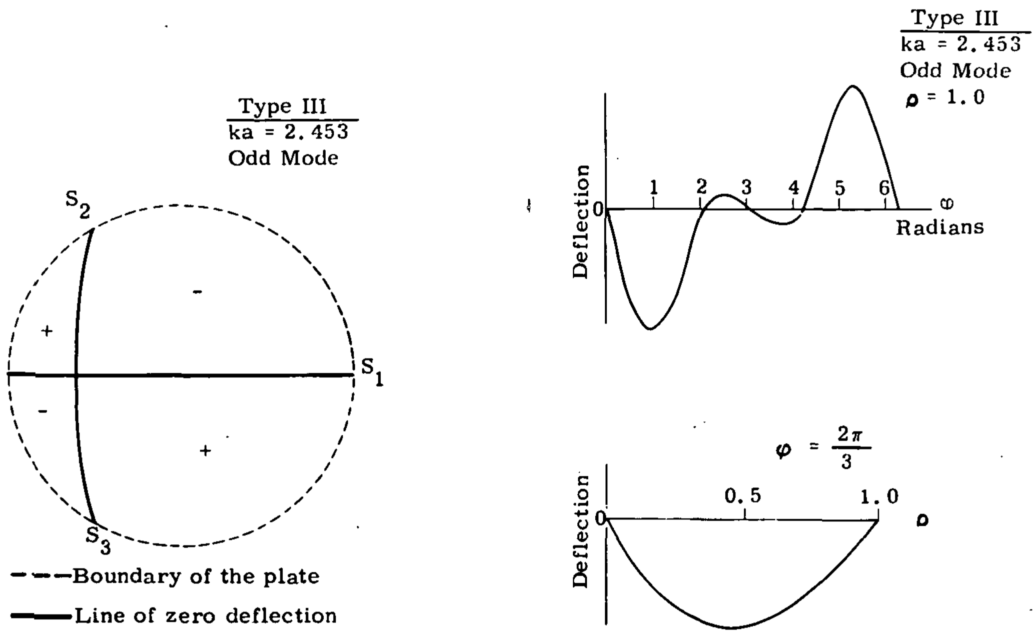


Figure 8. Mode Shape, Type III, ka = 2.453

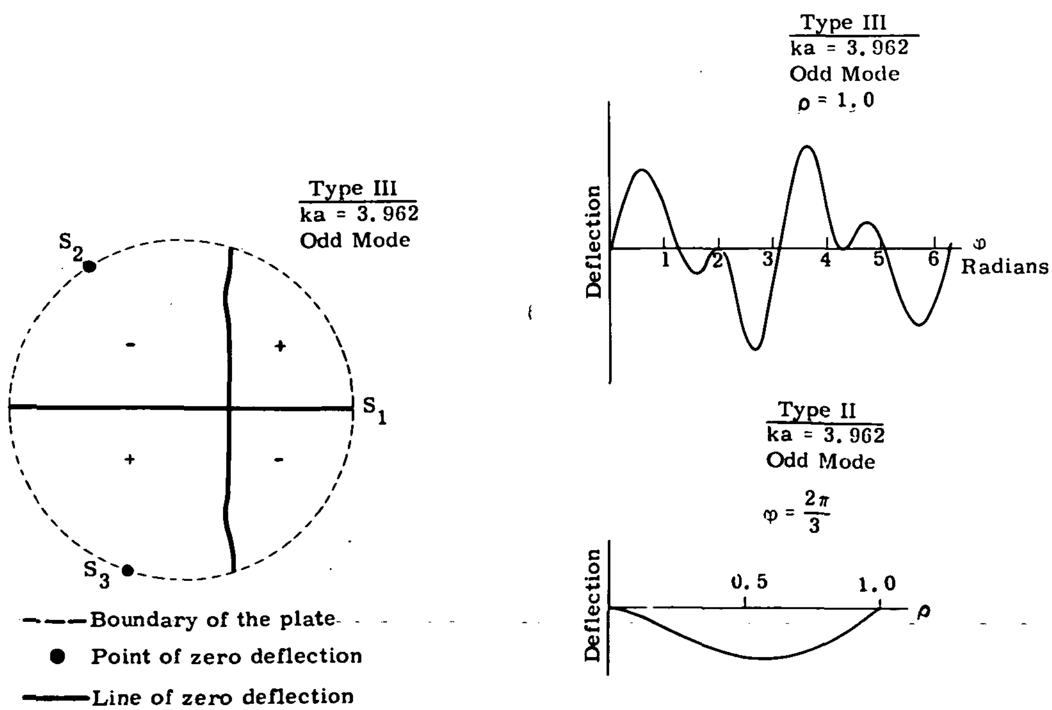


Figure 9. Mode Shape, Type III, ka = 3.962

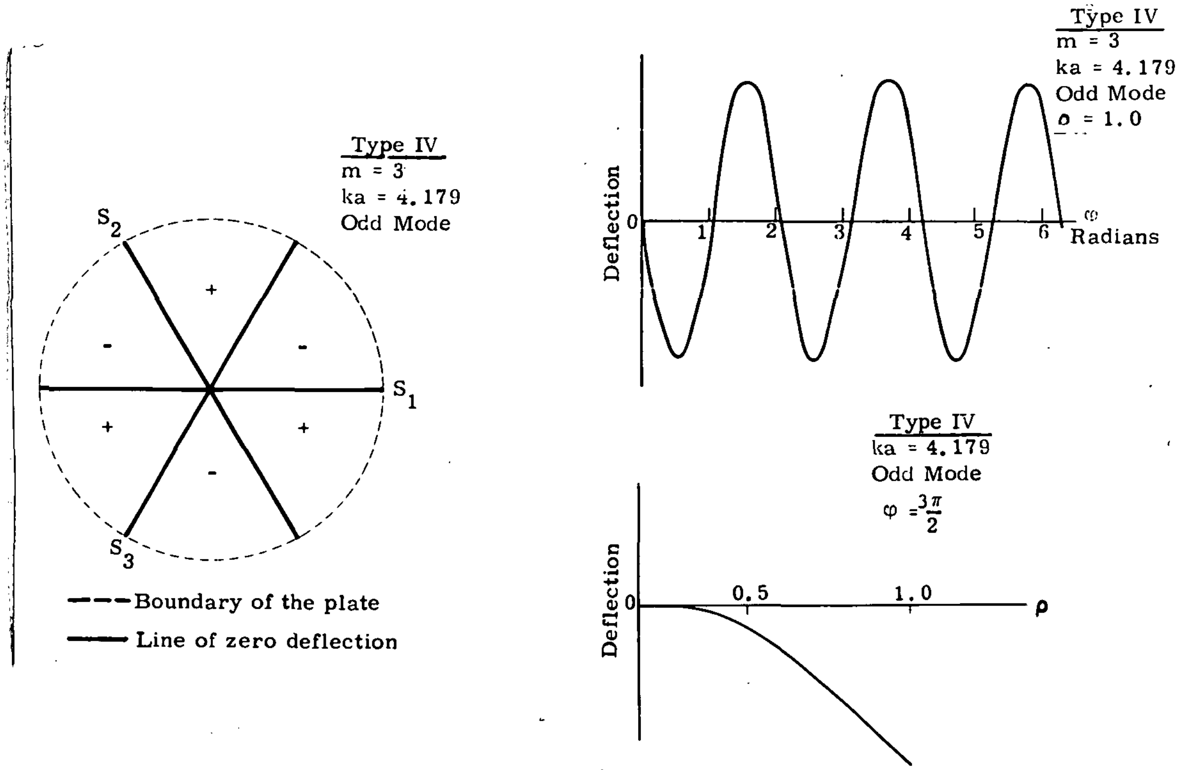


Figure 10. Mode Shape, Type IV, ka = 4.179

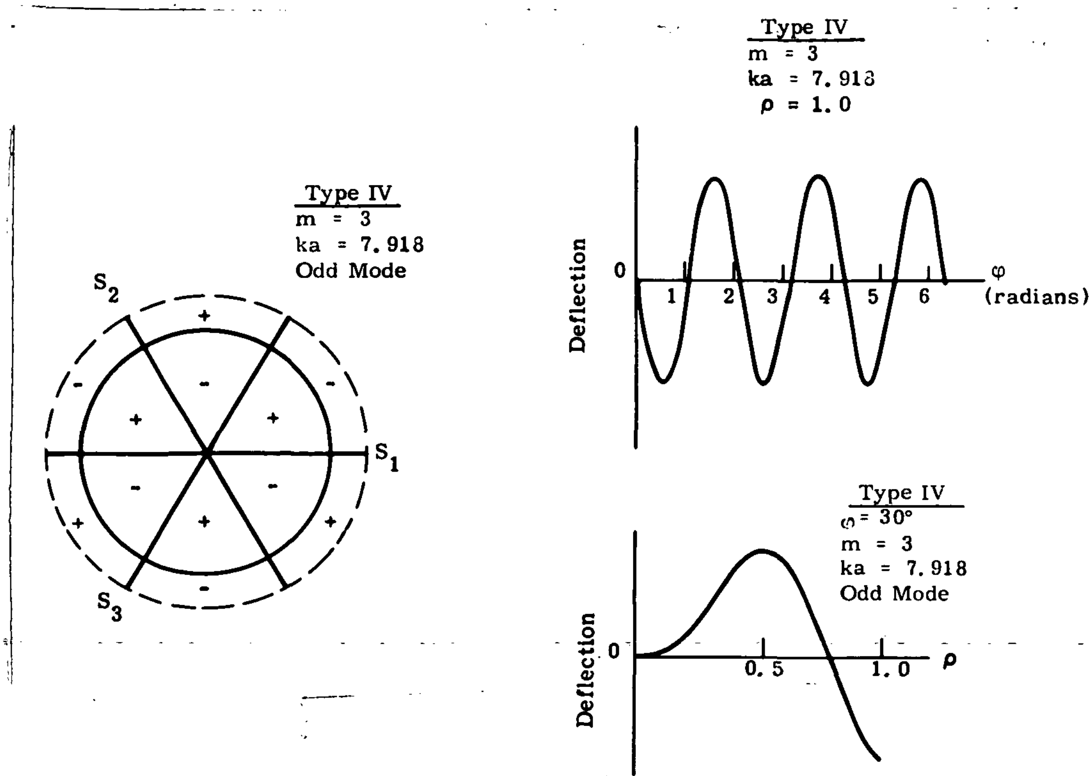


Figure 11. Mode Shape, Type IV, ka = 7.918

where ω = angular frequency in radians per second

$$D = \frac{Eh^3}{12\,(1-\nu^2)}$$

E = the modulus of elasticity

h = the thickness of the plate

ν = Poisson's ratio

μ = mass per unit area of the plate

and the normalized eigenvalue (k') is defined to be

$$k' = ka$$

where a is the radius of the plate. Thus, (k') is the eigenvalue (k) when the radius of the plate is unity.

The lines and points of zero deflection for each vibrational mode shown in Figures 4 through 11 are useful in identifying the particular mode in actual mirror vibration in the mode shape patterns.

The analysis shows that it is physically meaningful and convenient to group the resonant modes (eigenfunctions) into four types, and the important properties of each type of mode are listed in Table 2.

It is interesting to note that types I, II, and III exert force on the support points, while type IV does not. Consequently, types I, II, and III require the support points in order for them to exist. Type IV comprises the group of modes that are also members of vibrational modes of the free plate without support points; consequently, type IV does not require the support points in order for it to exist. As mentioned earlier, types I and II are even with respect to x-axis, while type III is odd. Type I, furthermore, possesses 120-degree rotational symmetry (120-degree rotational symmetry means that, when the mode pattern is rotated ±120 degrees, the pattern is unchanged). It is apparent that the low order modes are those of types I, II, and III, which are characteristic to the plate with the particular support arrangement, while the modes of type IV appear as high order modes. The pattern of the reaction forces by the support points can be deduced from the mode shape near the support points.

It was recognized in the early phase of the project that the presence of strong discontinuity in the boundary condition on the circumference of the plate (in the form of the delta function) heavily influences the final results of

TABLE 2. - PROPERTIES OF MODE TYPES I, II, III AND IV

TYPE I MODE
• Mode shape has 120° symmetry and is symmetrical with respect to x-axis • All three support points exert the same force simultaneously • Three support points are required in order to excite this type of mode
TYPE II MODE
• Mode shape is symmetrical with respect to x-axis • The support points exert force symmetrically with respect to x-axis • This is one of the degenerate modes (the frequency is the same as that of type III) • Three support points are required in order to excite this type of mode
TYPE III MODE
• Mode shape is antisymmetrical with respect to x-axis • The support points exert force antisymmetrically with respect to x-axis • This is one of the degenerate modes (the frequency is the same as that of type II) • Only two support points (S_2, S_3) are required in order to excite this type of mode
TYPE IV MODE
• Mode shape is antisymmetrical with respect to x-axis • The support points exert no force • This is a group of modes that are also free plate vibrational modes • No support points are required in order to excite this type of mode

the eigenvalues and eigenfunctions. This in turn required that careful consideration be given to the selection of the grid pattern because any error in the region of support points due to the irregular and crude approximation of the boundaries affects the overall accuracy of the computer solutions. The triangular grid pattern, which is frequently used and shown in Figure 12, does not provide a regular pattern along the circumference. The design effort to produce the appropriate grid pattern resulted in the polar grid pattern shown in Figure 13. The polar grid pattern is most natural and compatible to the circular geometry of the mirror structure.

Radial spacing between two adjacent grid circles has been so chosen as to make the aspect ratio of the quadrilateral elements small, resulting in a nearly square element. This is desirable for maintaining accuracy of the computer solution.

Let $\bar{r}_n$ be the radius of the n-th grid circle as shown in Figure 14. In terms of the outer diameter (D_1) and inner diameter (D_2) of the mirror plate, $\bar{r}_n$ is given by

$$\bar{r}_n = (\bar{c}_1 D_1 + c_2 D_2) / 2 \qquad (2)$$

where $\bar{c}_1$ and $\bar{c}_2$ are constants defined in Table 3. These constants have been so chosen that the resulting radial spacing between two successive grid circles keeps the aspect ratio of the plate elements in the neighborhood of unity. In other words, the deviations of the quadrilateral and triangular plate elements from the respective square and equilateral shapes are small as mentioned earlier.

In the central region of the mirror, the plate elements are approximately equilateral triangles. If the radial rays were extended to the center of the mirror, the plate elements in this region would be quadrilaterals of high aspect ratio and nearly wedge-shaped triangles, and it is desirable to avoid such shapes.

The grid point numbering system of Figure 13 called the machine oriented grid numbering system, creates a stiffness matrix with a narrow band and no active columns as used in NASTRAN. In general, the computation time is less when the chosen grid point sequence results in a narrow bandwidth and a small number of active columns. As a rule of thumb, the semibandwidth of a stiffness matrix is proportional to the maximum difference between any two connected grid point sequence numbers. (The semiband is defined as the maximum number of columns included from the diagonal term in any row to the most remote term inside the band. Columns of a matrix containing nonzero terms outside the band are referred to as "active columns".) For our sequence, this maximum difference is 18.

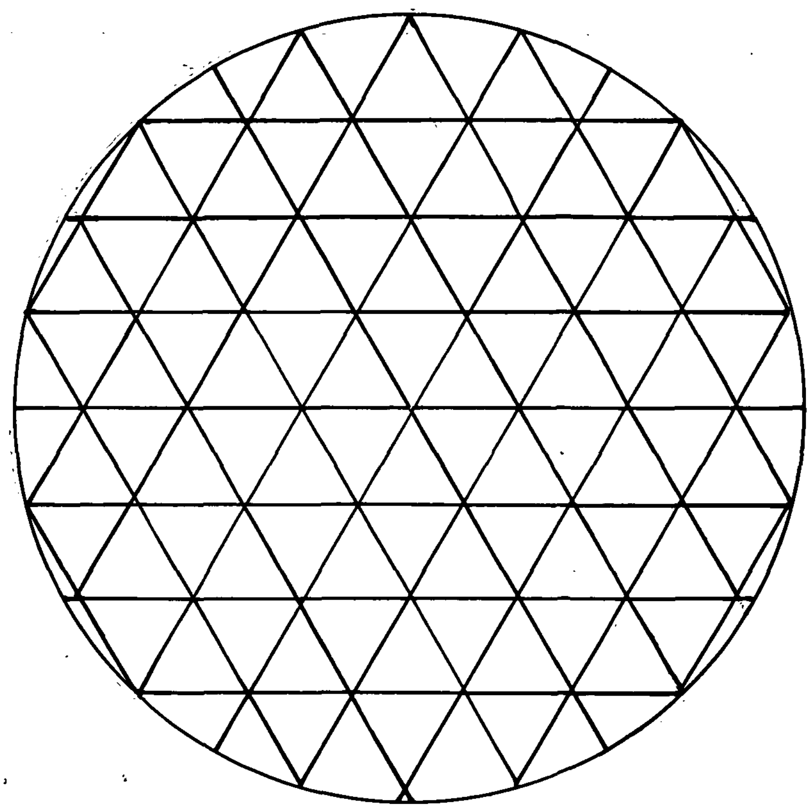
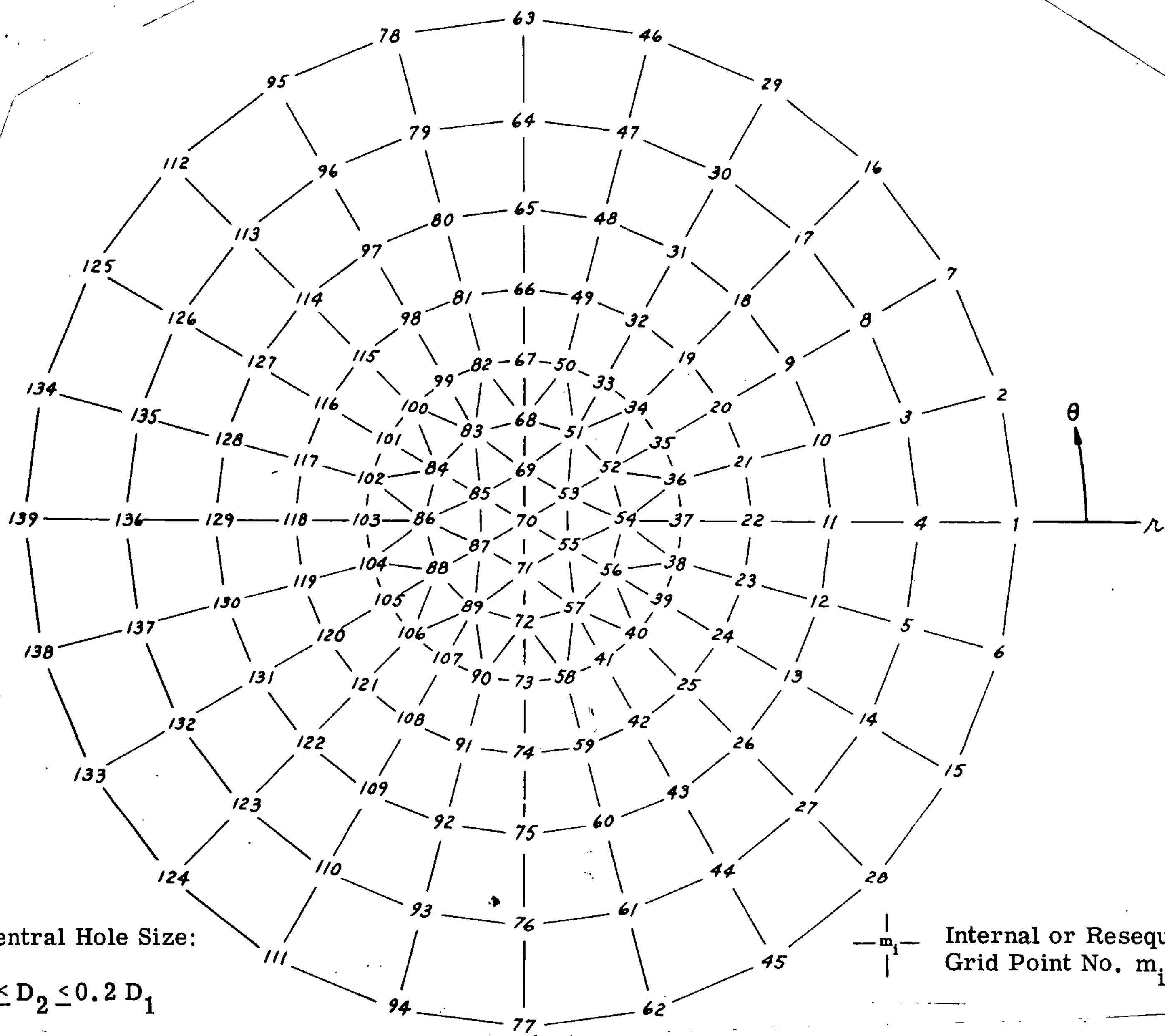


Figure 12. Triangular Grid Pattern



Central Hole Size:

$0 \leq D_2 \leq 0.2\, D_1$

Internal or Resequenced Grid Point No. m_i

Figure 13. Polar Grid Pattern and Grid Point Numbering System

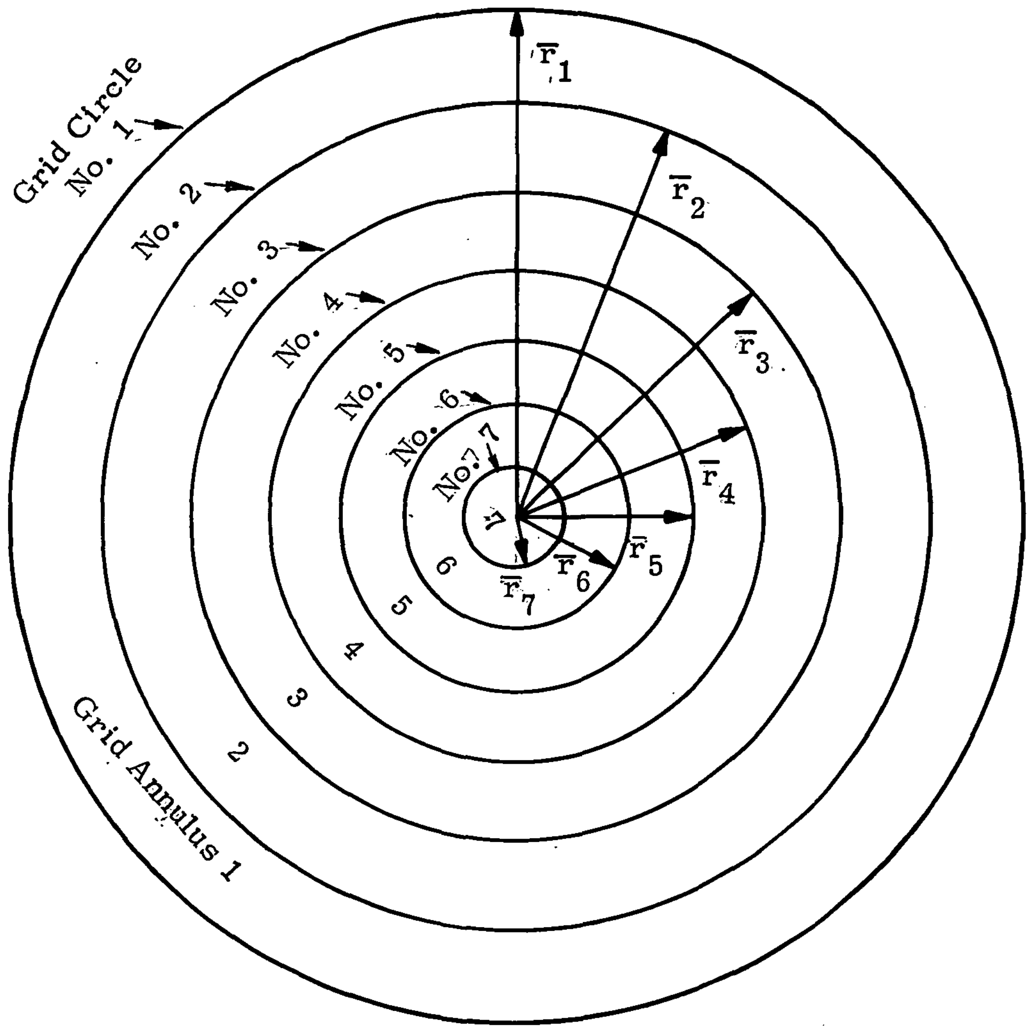


Figure 14. Mirror Grid Circles and Grid Annuli $r_8 = 0$ for the Center, Considered as Grid Circle No. 8

TABLE 3. - CONSTANTS C_1 AND C_2 OF EQUATION (2)

n	$\bar{r}_n$	Pattern 1 When $D_2 = 0$ (No Central Hole) (7 Annuli)		Pattern 2 $0 < D_2 \leq 0.15D_1$ (Central Hole Present) (6 Annuli)		Pattern 3 $0.15D_1 < D_2 \leq 0.2D_1$ (Central Hole Present) (5 Annuli)	
		c_1	c_2	c_1	c_2	c_1	c_2
1	$\bar{r}_1$	1.00	0	1.000	0	1.00	0
2	$\bar{r}_2$	0.80	0	0.781	0.219	0.76	0.24
3	$\bar{r}_3$	0.62	0	0.586	0.414	0.54	0.46
4	$\bar{r}_4$	0.46	0	0.412	0.588	0.34	0.66
5	$\bar{r}_5$	0.32	0	0.257	0.743	0.16	0.84
6	$\bar{r}_6$	0.20	0	0.120	0.880	0	1.00
7	$\bar{r}_7$	0.10	0	0	1.000	-	-
8	$\bar{r}_8$	0	0	-	-	-	-

A good foundation was thus established upon which the structural simulation system could be built: namely, the theoretical analysis and the physical insight into the structural behavior, the well-thought-out and carefully designed grid pattern and the grid point numbering system considered to be optimum. The project was then ready to proceed to computer use for the structural analysis.

The comparison of the normalized eigenvalues (k') obtained from the analysis and NASTRAN computation is shown in Table 4. (It is to be noted that the eigenvalues of the same mode shapes should be compared.) Due to the relationship of equation (1), the percentage of difference between k'_{theory} and $k'_{NASTRAN}$ in Table 4 is smaller than the percentage difference between the corresponding frequencies. The values of k' are indeed the appropriate quantity to compare because the solutions of the resonance equations produce k' values and the frequencies are the secondary quantity obtained from k'. The comparison of NASTRAN computation results for a flat plate having a 30-inch diameter and a 0.5-inch thickness, where $\nu = 0.2$, and that for a spherical plate having similar dimensions with a 178-inch radius of curvature is shown in Table 5. We note that the eigenvalues of the spherical plate are larger than those of the flat plate in the lower modes; the differences between the two cases diminish as the order of modes increases. This trend is expected when one notes that as the order of mode increases, the corresponding spatial period decreases; within the region of the spatial period of the higher order modes, the spherical plate is nearly flat.

The comparison of the mode shapes between the flat plate and the spherical plate described above shows that the nodal lines and the mode shapes are the same for all practical purposes. Sometimes, though not often, some mode shapes, especially the nodal lines, of the conjugate modes (types II and III) appear a little distorted. This occurs mostly because the computer outputs mode shapes that are linear combinations of the two conjugate modes having the same eigenvalue. Experience with various mode shapes indicates that the mode shapes are not strongly dependent on the exactness of the eigenvalues.

It was recognized early in the project that the eigenfunctions are dependent upon the characteristics of the backing plate, actuators, and support modes as well as the mirror plate. The Auxiliary Program was provided with the facility to find the structural response of the entire composite system, but only simple tests to check the sequence of the program were made.

Although the scope of the project is to produce the simulation software and does not include the task of using AOSS to obtain the results, it is still regrettable that some extensive testing was not performed because of the shortage of funding and time.

TABLE 4. - NORMALIZED EIGENVALUES OF THE FLAT CIRCULAR PLATE SIMPLY SUPPORTED AT THREE POINTS ON THE CIRCUMFERENCE ($\nu = 0.25$)

Mode Number	Type I Mode (120° Rotational Symmetry Mode)		Type II Mode (Conjugate Mode - Even)		Type III Mode (Conjugate Mode - Odd)		Type IV Mode (Member of Free Plate Modes)	
	k'_{theory}	$k'_{NASTRAN}$	k'_{theory}	$k'_{NASTRAN}$	k'_{theory}	$k'_{NASTRAN}$	k'_{theory}	$k'_{NASTRAN}$
1	2.385	1.86	2.453	1.87	2.453	1.875	4.179	3.54
2	3.289	3.11	3.962	3.61	3.962	3.613	7.918	
3	5.664		5.123	4.48	5.123	4.485		

TABLE 5. - COMPARISON OF THE COMPUTER CALCULATED VALUES OF THE EIGENVALUES FOR THE FLAT PLATE AND THE SHALLOW SPHERICAL PLATE WITH THREE SIMPLE SUPPORTS ALONG THE CIRCUMFERENCE

Mode Number	Type I Mode (120° Rotational Symmetry Mode)		Type II Mode (Conjugate Mode - Even)		Type III Mode (Conjugate Mode - Odd)		Type IV Mode (Member of Free Plate Modes)	
	Flat Plate $k'_{NASTRAN}$	Spherical Shell $k'_{NASTRAN}$	Flat Plate $k'_{NASTRAN}$	Spherical Shell $k'_{NASTRAN}$	Flat Plate $k'_{NASTRAN}$	Spherical Shell $k'_{NASTRAN}$	Flat Plate $k'_{NASTRAN}$	Spherical Shell $k'_{NASTRAN}$
1	1.86	1.97	1.87	1.875	1.875	1.876	3.54	3.55
2	3.11	3.49	3.61	3.665	3.613	3.665		
3			4.48	4.62	4.485	4.62		

A considerable effort has been devoted to investigating situations of matrix ill-conditioning. Such situations occur when the matrix is approaching its null condition and the inversion of the matrix is computed. The inversion of the ill-conditioned matrix is very sensitive to the errors in the numerical solutions and therefore contains a large error.

Ill-conditioning of the stiffness matrix can occur in a composite structure system when the mirror plate and the backing plate are coupled because of a set of weak springs. Therefore, there is a range of parameters for which the present method of the computer solution is valid; however, the limit of this range has not been determined in this project. Matrix ill-conditioning also occurs during plate element rotation when the in-plane component is not properly constrained.

The machine-oriented grid numbering system described earlier causes a great deal of confusion and aggravation when the user tries to study the printed output of the computer solutions because the grid point numbers are not arranged in an orderly manner. Therefore, a new user-oriented grid numbering system that is very easy to use has been devised. The Auxiliary Program executes the computation in the machine-oriented grid system and outputs the results in the user-oriented grid numbering system.

Table 6 is included as an illustration of the structural parameters with which the Auxiliary Program is concerned.

The entire output of the Auxiliary Program is received by the preprocessing program before they are used by the Active Optics Simulation Program (AOSP). The preprocessing program is an interface between the NASTRAN operation and AOSP, and its role depends largely on the individual computer model and computer facility. It is hoped that as AOSS is used in different computer facilities the preprocessing program will absorb most of the specific modifications and leave the Auxiliary Program and AOSP intact as much as possible. Therefore, the preprocessing program that is specifically designed for LRC has not been implemented.

TABLE 6. - AUXILIARY PROGRAM STRUCTURAL PARAMETERS

1. Mirror - Backing Plate Module

   (a) Mirror material - See Table 1

   (b) Mirror diameter:

   range: 0.76-3.05 meters (30-120 inches)
   nominal: 0.76 meters (30 inches)

   (c) f-number:

   range: 1.5-5
   nominal: 2 29/30 (radius of curvature of nominal 0.762-meter mirror is 4.5212 meters)

   (d) Mirror diameter-to-thickness ratio:

   range: 10-80
   nominal: 60

   (e) Mirror shape:

   candidates: spherical, aspheric
   nominal: spherical

   (f) Size of central hole:

   range: 0 - 20 percent of mirror diameter
   nominal: 0

   (g) Backing plate material: Case aluminum

   (h) Backing plate thickness: 0.019 meter

   (i) Number of actuator holes: 58

2. Mirror Mount Submodule

   There are six types of mirror mount options:

   - simply supported
   - three-point "kinematic" mount
   - "A" frame links
   - Tangent bars
   - Axial flexural leaf springs
   - General purpose elastic mount

Mirror can be supported at points along the central hole, about circumference of mirror, or both.

ACTIVE OPTICS SIMULATION PROGRAM (AOSP)

The AOSP receives input data from the user and structural data computed by NASTRAN from the preprocessing program. It simulates the entire mirror surface control system (called the active optics). All parameters are accessible to the user as the real time simulation proceeds; therefore, the use can monitor those outputs of interest as he changes the various parameters in the simulation system. This will be the usual mode of operation as the user evaluates a particular system or optimizes an already selected system configuration.

Because the structural solution is provided by the Auxiliary Program the size of AOSP is small enough to provide a quick turn-around time; and, when the interactive mode of operation is implemented in the future, AOSP can be easily adapted to that mode of operation.

Some of the most important program modules are described in the following paragraphs.

Interpolation Module

The computer solutions are obtained at a set of grid points, and whenever the values between the grid points are needed they must somehow be computed from the values already available at the grid points in accordance with a particular procedure. This is called the interpolation and it is needed nearly always when computer solutions are made.

The interpolation module of AOSP provides the interpolated values between the grid points anywhere over the mirror surface; and, therefore, does not represent any hardware components or physical processes but is a mathematical operation.

It is to be noted that the accuracy and the quality of the simulated results are as good as the interpolation scheme used. A reliable interpolation technique is then required to fine the values between the grid points, and the bicubic spline fit interpolation technique is suitable for this purpose. When the system geometry is circular, as is the primary mirror, the currently available bicubic spline fit in a rectangular coordinate system (DeBoor, 1962 and Vogl, 1971) (Refs. 15 and 16) is not conveniently adaptable for the circular boundary region. This fact is especially critical when the system has highly discontinuous boundary conditions such as the support point around the rim of the circular mirror. Furthermore, the choice of the polar grid pattern of Figure 13 over the pattern of Figure 12 requires that a new interpolation scheme be developed.

An interpolation method, called the curvilinear bicubic spline fit interplation scheme, has been developed specifically for AOSP. It is compatible with the circular geometry and results in savings in computer time, reduced programming complexity, and improved quality of the interpolation. (See Refs. 6 and 13.) This new interpolation scheme contains a physically significant feature that originates from the observation that the physical laws are formulated with length as the basic unit and the differential equations describing physical behavior consist of the differentiation with respect to a distance.

There are two approaches that can be taken when developing the interpolation function in the polar coordinate system; one approach is to express the interpolation function in terms of $\bar{r}$ and θ, as shown in Figure 15(a), and the other is in terms of $\bar{r}$ and $\bar{r}\theta$, as shown in Figure 15(b). In most applications involving the physical system, the latter formulation is more meaningful because it uses the arc length ($\bar{r}\theta$) as the basis of interpolation and implicitly takes the patch shape to be that shown in Figure 15(b) instead of that shown in Figure 15(a). The latter approach, which is used here, takes account of the facts that, within a patch, the arc length between the two radial lines becomes larger as its radial distance from the center increases, and the differentiation of function with respect to arc length ($\bar{r}\theta$) instead of angle (θ) is physically a more meaningful quantity. These considerations are in marked contrast to the former approach, which essentially substitutes r, θ variables for x, y variables of the rectangular grid pattern.

The isometric drawing of Figure 16 shows a displaced hemisphere, which is based on the interpolated values between the polar grid points. The isometric plotter moves in a straight line in a rectangular coordinate system and plots the interpolated values of the function at given intervals. Each interpolated value was computed using the curvilinear bicubic spline fit formulae and was then supplied to the plotting machine. The isometric drawing serves to illustrate the working of the interpolation on a qualitative basis. In Table 7, the interpolated values are compared with the actual values of the displaced hemisphere function.

Control Law Module

AOSP simulates the modal control law. This technique has the promise of achieving the required mirror control performance by using a small number of actuators, and it is most effective in situations where the actual configuration calls for only a few actuators. The AOSP control law module follows the modal control law technique outlined in reference 3.

The control law module contains a control compensator submodule that provides two sets of lag-lead (or lead-lag) networks and a gain adjustment for each mode channel. The user can vary the number of controlled modes,

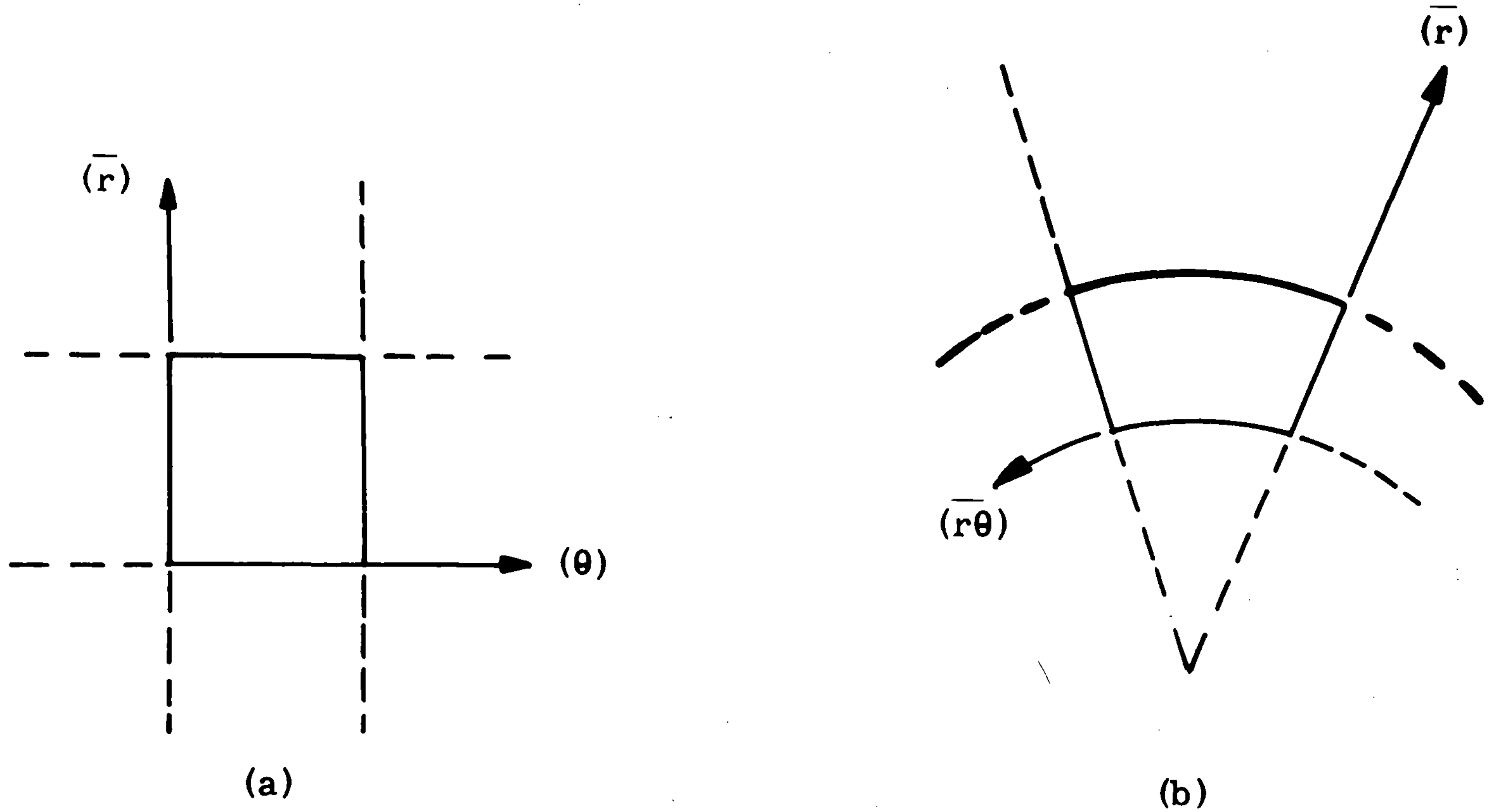


Figure 15. Comparison of Patch Shapes

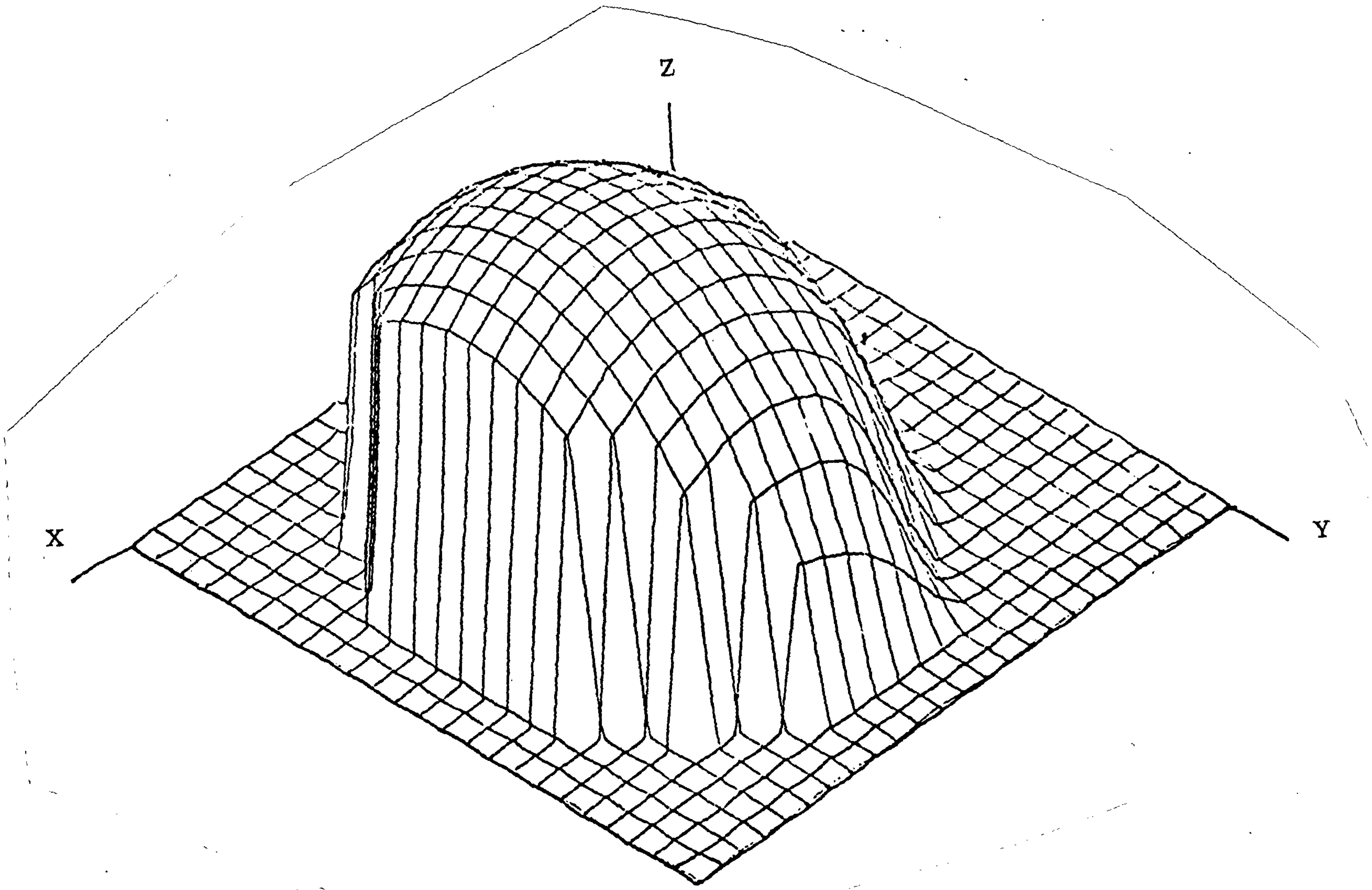


Figure 16. Isometric Drawing of a Hemisphere that is Displaced 40% of the Radius in the X-Direction

TABLE 7. - COMPARISON OF INTERPOLATED VALUES WITH THE ACTUAL VALUES OF THE TEST FUNCTION

$$z = \sqrt{25 - (x - 2.5)^2 - y^2} \quad ; \quad z \geq 0$$

$$r^2 = x^2 + y^2$$

$$\theta = \tan^{-1} \frac{y}{x}$$

r	θ (deg.)	Value of the Function	Interpolated Value
0.5	7	4.5805420	4.60235
0.5	60	4.44440973	4.4350625
0.5	90	4.3011628	4.2978738
0.5	180	4.0000000	4.0182309
0.5	300	4.4440968	4.3506255
0.5	359	4.5825341	4.6117449
4.9	7	4.3654760	4.3405932
4.9	62	2.4984117	2.3309577
4.9	91	0.00	0.0831669
4.9	181	0.00	-0.0178549
4.9	302	2.7790261	2.6518625
4.9	359	4.3859170	4.3525210

actuator locations, the total number of the modes to be considered, and the amount and break frequencies of the compensators.

It was recognized early in the project that the matrix inversion operations in the control law module can be ill-conditioned when the figure sensor points are placed on the nodal lines of a mode. Therefore, various approaches to test the degree of ill-conditioning have been investigated. Although it would be highly desirable to include these tests, the funding and schedule constraints did not allow their implementation. At present, AOSS computes only the determinant of the matrix and the user may be warned when the determinant becomes unusually small. The precautionary test of checking the accuracy of the matrix inversion has not been implemented at this time.

The control law module consists of many submodules, providing the user has access to any variables of interest without difficulty.

The immediate use of AOSP will probably occur when the modal control law is investigated to determine the number of actuators needed, the optimum locations of the actuators, the maximum number of modes that AOSP must work with, etc. Subsequently, AOSP will be useful to optimize the parameters in a given requirement and disturbance.

It is expected that in the future there will be come modifications to the modal control law as well as additions of other parameter variations to the program. Because of the modular structure of AOSS, these modifications can be easily accommodated.

Actuator Module

The force actuator is simulated by the actuator module (see Ref. 1). The simulation includes the effect of backlash, dead zone, the saturation of the electronics, and electronic time constants.

The typical design goals are:

(a) Resolution: 2.54 - 12.7 nanometers

(b) Degree of Backlash: 12.7 nanometers (This has been designed but has not been implemented.)

(c) Dynamic Range: 0 - 0.05 meters

(d) Slewing Rate: 0.076 - 2.54 micrometers per second

(e) Force: 4.45 - 445 newtons (1 - 100 pounds)

(f) Spring Constant: 175 - 175, 000 newtons per meter
(1 - 1000 pounds per inch)

The block diagram of the force actuator module is shown in Figure 17.

Mirror Module

The mirror module, being a part of the mechanical structure, has been solved previously by NASTRAN through the use of the auxiliary program, and all the structural information of the mirror plate is included in the eigenvalues and eigenfunctions. The mirror module computes the mode amplitude of the mirror surface error. The mirror module allows the user to specify the damping constant of each mode. The transfer function of the mirror plate in each mode is a second-order system,

$$\frac{C_i}{D_i} = \left[\frac{1}{s^2 + \zeta_i s + \lambda_i} \right]$$

where C_i is the amplitude deflection of the i-th mode

D_i is the force amplitude of i-th mode

ζ_i, λ_i are constants

s is the Laplace transform variable

Disturbance Module

The disturbance module simulates the mechanical force disturbance over the entire region of the mirror or over one portion of the mirror surface. Using this module, the mirror surface deformation due to an applied force can be obtained, and this information is often very useful. The disturbance due to the inertial force and the thermally induced disturbance can also be simulated. This module requires more work in the area of the user input program so that disturbance can be specified in simple mathematical expressions. Also more testing is necessary for this module.

Figure Sensor Module

The figure sensor module simulates the moire pattern figure sensor system (Ref. 2). This figure sensor is capable of measuring the mirror surface

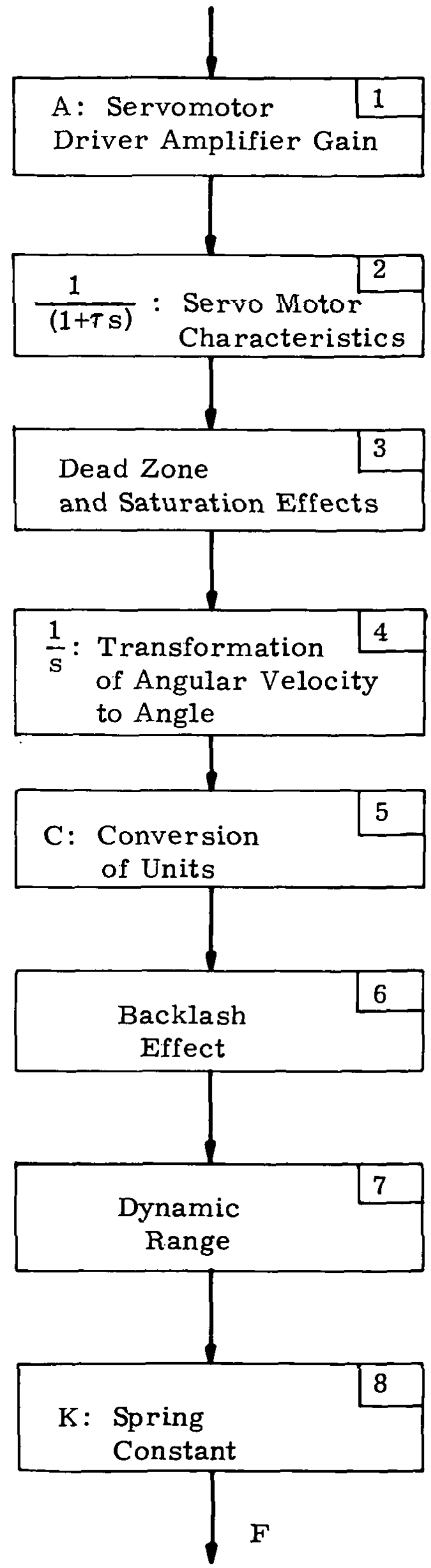


Figure 17. Force Actuator Module

deformation of an aspherical mirror. The simulation includes the effect of misalignment of the figure sensor and electronic signal processing.

The electronic system is provided with two time constants in each channel, and the transfer function becomes of the form,

$$\frac{A}{(1+\tau_1 s)\,(1+\tau_2 s)}$$

where A, τ_1, τ_2 are constants

s is the Laplace transform variable

It is to be noted that the operation of the moire figure sensor depends upon the diffraction effect of the moire master. Therefore, the ordinary ray-tracing program will not describe the complete behavior of the figure sensor. Furthermore, any attempt to use a ray-tracing program is not wise because it will increase the computer time by a hugh proportion. A complete ray-tracing program must be performed four times for each time increment of the transient behavior and each ray tracing has many rays. Therefore, it was necessary to develop a new mathematical formulation more suitable to AOSP; and, eventually, the "quasi-optics" formulation was developed (Ref. 6).

The quasi-optics formulation requires much less computer time and takes into account both the geometric optics and the diffraction effects that include the spatial filter of the finite size lens and the edge effect of the moire master pattern.

Mathematically, it is an asymptotic expansion of the electromagnetic wave in the high frequency range.

Even with the quasi-optics formulation, the total computer time will be needlessly large. At present, one recommended procedure is to run AOSP with the optical part of the figure sensor considered to be an ideal system. A great deal of investigation can be accomplished in this mode of operation.

The quasi-optics formulation program can then be operated separately to examine the behavior of the optical part of the figure sensor. When all the necessary investigations are performed separately, the quasi-optics formulation can be inserted into the loop of AOSP.

The simulation of the electronics system in the figure sensor has been implemented; however, the optical part of the figure sensor module remains unimplemented.

Although the quasi-optics formulation is not new, it is not used frequently in this type of work and therefore needs more testing.

Transient Analysis

The 4th order Runge-Kutta method is used for the transient analysis of AOSP. It is a self-starting method and is stable if a small enough integration time increment is chosen. The use of this method also results in a simpler program structure.

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